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# APPROXIMATING THE CHARACTERISTICS OF SEQUENTIAL TESTS

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*Abstract.* In this paper the dependence structure inherent in the sequence of partial sums is utilized to derive accurate approximations for the tail probabilities of the stopping time associated with sequential tests for the normal mean. The approximations for these tail probabilities are used to approximate the overall significance level, power function, expected stopping time, and the variance of the stopping time associated with the sequential tests discussed here. Moreover, after the testing procedure has been completed, the approximations derived in this paper are used to evaluate *P*-values and confidence intervals for the normal mean. Numerical results are presented for the sequential probability ratio test and the asymptotically optimal Bayes test.

## 1. INTRODUCTION

The standard approach in studying the characteristics of sequential tests for the normal mean is based on asymptotic results for boundary crossing probabilities of partial sums, employing martingale and renewal theory methods (see [16], [21], [26], and [29]). Glaz and Johnson [9] introduced a new approach for approximating boundary crossing probabilities for partial sums of independent normal observations. These results were utilized in approximating various characteristics of sequential tests for the normal mean incuding: the overall significance level, power function, tail probabilities, expected stopping time, and the variance of the stopping time associated with the sequential test. The approximations in [9] are remarkably accurate in the case when an early termination of the test is likely. For sequential tests that have a moderate or long termination time, more accurate approximations are needed. In this paper we address this need. Moreover, after the testing procedure has been completed we use the improved approximations to evaluate *P*-values and confidence intervals for the mean.

To fix ideas, let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed (i.i.d.) observations from a normal distribution with unknown mean $\theta$ and known variance $\sigma^2$. In what follows without loss of generality we will assume that $\sigma^2 = 1$. We are interested in testing

$$H_0\colon \theta \leqslant \theta_0 \text{ vs. } H_1\colon \theta > \theta_0.$$

In what follows without loss of generality we will assume that $\theta_0 = 0$. The sequential testing procedures considered in this paper are of the following general form: take observations $\{X_i\}_{i=1}^{\infty}$ sequentially. At the $n$-th stage *stop and reject* $H_0$ if

$$S_n = \sum_{i=1}^{n} X_i \geqslant b_n, \tag{1.1}$$

*stop and accept* $H_0$ if

$$S_n \leqslant a_n,$$

and *continue sampling* by taking another observation if

$$S_n \in I_n = (a_n, b_n). \tag{1.2}$$

Let

$$\tau = \inf\{n \geqslant 1;\ S_n \notin I_n\} \tag{1.3}$$

be the random *stopping time* associated with the sequential test. For $\theta > 0$ the *power function* of the sequential test is given by

$$\beta = \sum_{n=1}^{\infty} P_\theta\{(\tau > n-1) \cap (S_n > b_n)\} = \sum_{n=1}^{\infty} P_\theta\{[\bigcap_{j=1}^{n-1} (S_j \in I_j)] \cap (S_n > b_n)\}. \tag{1.4}$$

The sequence of intervals $\{I_n\}_{n=1}^{\infty}$ defines the continuation region for the sequential test and is determined by selecting a parameter $\theta_1 > 0$ so that

$$\beta(0) = \alpha \quad \text{and} \quad \beta(\theta_1) = 1-\alpha,$$

where $\alpha$ is the desired *overall significance* level of the test.

The most common characteristics of sequential tests that are of interest include: the tail probabilities

$$P_\theta(\tau > n) = P_\theta\{\bigcap_{j=1}^{n} (S_j \in I_j)\}, \tag{1.5}$$

the average sample size

$$\mathrm{E}_\theta(\tau) = \sum_{n=1}^{\infty} P_\theta(\tau > n), \tag{1.6}$$

and the variance of the sample size needed to carry out the testing procedure

$$\mathrm{Var}_\theta(\tau) = 2\sum_{n=1}^{\infty} nP_\theta(\tau > n) + \mathrm{E}_\theta(\tau)[1-\mathrm{E}_\theta(\tau)]. \tag{1.7}$$

After the completion of the testing procedure it is often of interest to evaluate the $P$-value for the sequential test and to construct a confidence interval for $\theta$.

In Section 2 of this paper we derive accurate approximations for the characteristics of sequential tests, emphasizing the evaluation of $P$-values and confidence intervals for $\theta$. Numerical methods used in computing the approximations are briefly outlined in Section 3. In Section 4 we particularize to two well-established sequential tests: sequential probability ratio test [25], [29], and asymptotically optimal Bayes sequential test [2], [28]. Numerical results are presented for each of these tests and are evaluated by an extensive simulation study. A discussion of the numerical results is presented at the end of the paper.

## 2. APPROXIMATIONS FOR SEQUENTIAL TESTS

**2.1. Preliminary results.** Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. normal observations with mean $\theta$ and variance $\sigma^2 = 1$. The approximations derived in this section have their roots in the approximation for $P(\tau > n)$. The following concept of positive dependence plays an important role.

A nonnegative function of two variables, $f(x, y)$, is said to be *totally positive of order two*, $TP_2$ (cf. [11]), if for all $x_1 < x_2$ and $y_1 < y_2$

$$f(x_1, y_1)f(x_2, y_2) - f(x_1, y_2)f(x_2, y_1) \geqslant 0.$$

A nonnegative real-valued function of $n$ variables, $f(x_1, \ldots, x_n)$, is said to be *multivariate totally positive of order two*, $MTP_2$ (cf. [12]), if for any $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\boldsymbol{R}^n$

$$f(\boldsymbol{x} \vee \boldsymbol{y})f(\boldsymbol{x} \wedge \boldsymbol{y}) \geqslant f(\boldsymbol{x})f(\boldsymbol{y}),$$

where

$$\boldsymbol{x} \vee \boldsymbol{y} = (\max(x_1, y_1), \ldots, \max(x_n, y_n)) \quad \text{and} \quad \boldsymbol{x} \wedge \boldsymbol{y} = (\min(x_1, y_1), \ldots, \min(x_n, y_n)).$$

A sequence of random variables $X_1, \ldots, X_n$ is said to be $MTP_2$ if its joint density function is $MTP_2$.

Remark. Barlow and Proschan [1] define a related concept of positive dependence: $TP_2$ *in pairs*. If the support for the distribution of $X_1, \ldots, X_n$ is a product space, then the $TP_2$ in pairs is equivalent to $MTP_2$ (see [3]).

It follows from Theorem 2.1 of [9] that the sequence of partial sums $\{S_i\}_{i=1}^{n}$ defined in equation (1.1) is $MTP_2$ for any $n \geqslant 1$. Therefore, for the problem at hand $\{|S_i|\}_{i=1}^{n}$ is $MTP_2$ for any $n \geqslant 1$ ([13], Theorem 3.1) and the product-type inequalities that have been introduced in Theorem 2.3 of [8] can be utilized in approximating $P(\tau > n)$ and other characteristics of sequential tests.

**2.2. Approximations for $P(\tau > n)$, $E(\tau)$, $\mathrm{Var}(\tau)$ and $\beta(\theta)$.** For $1 \leqslant m \leqslant n$ it follows from Theorem 2.3 of [8] and equation (1.5) that

$$P(\tau > n) \geqslant \gamma_{m,n}, \tag{2.1}$$

where

$$\gamma_{m,n} = \eta_{1,m} \prod_{k=m+1}^{n} \frac{\eta_{k-m+1,k}}{\eta_{k-m+1,k-1}} \tag{2.2}$$

and

$$\eta_{i,j} = \begin{cases} 1, & i > j, \\ P\{\bigcap_{t=i}^{j} (S_t \in I_t)\}, & 1 \leqslant i < j \leqslant n. \end{cases} \tag{2.3}$$

In what follows we abbreviate $\gamma_{m,n}$ to $\gamma_m$. In [9], $\gamma_m$ was evaluated for $1 \leqslant m \leqslant 3$ only. It was noted in [9] that, for sequential tests possessing moderate to long termination times, a more accurate approximation for $P(\tau > n)$ is needed. Since $\{\gamma_m\}_{m=1}^{n}$ is an increasing sequence, a more accurate approximation can be obtained for higher values of $m$. In Section 3 we present a new approach that will enable us to evaluate $\gamma_m$ for $1 \leqslant m \leqslant 7$. In Section 4 we compare our best approximation $\gamma_7$ with the approximations derived in [9].

To evaluate approximations for $\mathrm{E}(\tau)$ and $\mathrm{Var}(\tau)$ we employ equations (1.6) and (1.7), respectively, and approximate the terms $P(\tau > n)$ in these equations by $\gamma_7$. In Section 4 we compare these approximations with the approximations that are based on approximating $P(\tau > n)$ by $\gamma_3$.

It follows from equation (1.4) that

$$\beta(\theta) = \sum_{n=1}^{\infty} \beta_n(\theta, b_n), \tag{2.4}$$

where

$$\beta_n(\theta, x) = P\{[\bigcap_{j=1}^{n-1} (S_j \in I_j)] \cap (S_n > x)\}. \tag{2.5}$$

To approximate the power function we approximate the terms $\beta_n(\theta, x)$ by

$$\gamma_{7,n}^{*} = \gamma_{7,n-1}\, \eta_{n-6,n}^{*} / \eta_{n-6,n-1}, \tag{2.6}$$

where $\gamma_{7,n-1}$ and $\eta_{n-6,n-1}$ are defined in (2.2) and (2.3), respectively, and for $i < n-1$

$$\eta_{i,n}^{*} = P\{[\bigcap_{j=i}^{n-1} (S_j \in I_j)] \cap (S_n > b_n)\}. \tag{2.7}$$

In Section 3 we outline the method for evaluating $\eta_{n-6,n}^{*}$ and $\gamma_{7,n}$. In Section 4 we compare this new approximation with the approximations that have been studied in [9].

**2.3. *P*-values for sequential tests.** In what follows we adopt the definition of $P$-values from [17], where a detailed discussion about the rational behind that definition is presented. For sequential tests considered in this paper it follows from equations (2)–(5) of [17] that if the test terminates at stage $k$ with a reject $H_0$ decision and an observed value of $S_k = s_R \geqslant b_k$, then

$$P\text{-value} = \sum_{n=1}^{k-1} \beta_n(0, b_n) + \beta_k(0, s_R), \tag{2.8}$$

where $\beta_n(\theta, x)$ is defined in (2.5). If the test terminates at stage $k$ with an accept $H_0$ decision and an observed value of $S_k = s_A \leqslant a_k$, then

$$\text{(2.9)} \qquad P\text{-value} = 1 - \Big[\sum_{n=1}^{k-1} \alpha_n(0, a_n) + \alpha_k(0, s_A)\Big],$$

where for $n \geqslant 1$

$$\text{(2.10)} \qquad \alpha_n(\theta, x) = P\Big\{\Big[\bigcap_{j=1}^{n-1} (S_j \in I_j)\Big] \cap (S_n < x)\Big\} = P(\tau > n-1) - \beta_n(\theta, x).$$

Remarks. 1. It is routine to verify that if the $P$-values are defined as above, then when $H_0$ is true the distribution of the $P$-values is uniform over the interval $[0, 1]$.

2. The overall significance level of the test is given by

$$\text{(2.11)} \qquad \alpha = \sum_{n=1}^{\infty} \beta_n(0, b_n).$$

The evaluation of the $P$-value and the overall significance level of the tests are based on the algorithms for approximating $P(\tau > n)$ and $\beta_n(\theta, x)$ that are given in Section 3. In Section 4 we evaluate our most accurate approximations for the $P$-values and the overall significance levels based on $\gamma_7$ and $\gamma_7^*$, given in (2.2) and (2.6), respectively.

**2.4. Confidence intervals for the normal mean following a sequential test.** After a sequential testing procedure has been completed it is often of interest to present a confidence interval for the population mean $\theta$. The problem of constructing confidence intervals following a sequential test has been studied mainly for group sequential tests with at most five stages (see [5], [14], [18], and [24]). In this case iterative numerical algorithms have been used to construct the desired confidence intervals. For a general sequential test, asymptotic results have been utilized to construct confidence intervals following the testing procedure (see [20], [21], [23], [26], [29], [30]). For the special case of the exponential distribution Bryant and Schmee [4] and Madsen and Fairbanks [17] discuss numerical algorithms for constructing confidence intervals for the mean following a sequential probability ratio test [25]. Their confidence intervals use the well-known relation between confidence sets and test of hypothesis as outlined in Section 9.2 of [29]. The reason for adapting this approach rather than utilizing the maximum likelihood estimator of $\theta$, which for the problem at hand is based on $S_\tau$, is the presence of a sizable bias ([29], Chapter 9).

For the sequential tests considered in this paper (see equations (1.1)–(3.1)) we employ the approach in [17] and utilize the approximations derived in our paper to construct approximate confidence intervals for $\theta$ and the median unbiased estimator of $\theta$. Assume that the sequential test terminates at stage $k$

with a reject $H_0$ decision and an observed value of $S_k = s_R \geqslant b_k$. Then a $(1-2\alpha)\,100\%$ confidence interval for $\theta$, denoted by $(\theta_{RL}, \theta_{RU})$, is obtained from the equations

$$\sum_{j=1}^{k-1} \beta_j(\theta_{RL}, b_j) + \beta_k(\theta_{RL}, s_R) = \alpha \tag{2.12}$$

and

$$\sum_{j=1}^{k-1} \beta_j(\theta_{RU}, b_j) + \beta_k(\theta_{RU}, s_R) = 1-\alpha, \tag{2.13}$$

where $\beta_j(\theta, x)$ is given in (2.5) and $0 < \alpha < 0.5$ is a specified quantity. Now, suppose that the test terminates at stage $k$ with accept $H_0$ decision and an observed value of $S_k = s_A \leqslant a_k$. In this case, a $(1-2\alpha)\,100\%$ confidence interval for $\theta$, denoted by $(\theta_{AL}, \theta_{AU})$, results from the solution of the equations

$$\sum_{j=1}^{k-1} \alpha_j(\theta_{AL}, a_j) + \alpha_k(\theta_{AL}, s_A) = 1-\alpha \tag{2.14}$$

and

$$\sum_{j=1}^{k-1} \alpha_j(\theta_{AU}, a_j) + \alpha_k(\theta_{AU}, s_A) = \alpha, \tag{2.15}$$

where $\alpha_j(\theta, x)$ is given in (2.10). The numerical algorithms used in evaluating the confidence intervals $(\theta_{RL}, \theta_{RU})$ and $(\theta_{AL}, \theta_{AU})$ are described in Section 3.

Along with the confidence intervals for $\theta$ it is often useful to present the approximate median unbiased estimate of $\theta$. This estimate is obtained by solving equations (2.12) and (2.14) for $\alpha = 0.5$. The approximate median unbiased estimators obtained from (2.12) and (2.14) are denoted by $\theta_{RM}$ and $\theta_{AM}$, respectively.

In Section 4 we evaluate the approximate median unbiased estimates and confidence intervals for $\theta$ for selected values of $\alpha, k, s_R$ and $s_A$. These approximations are based on the accurate approximations for $P(\tau > n)$ (equations (2.1)–(2.3)) developed in this paper.

## 3. NUMERICAL ALGORITHMS FOR EVALUATING THE APPROXIMATIONS

In this section we present a brief outline for the numerical procedures used to evaluate the approximations discussed in Section 2. To evaluate $\gamma_{m,n}$, the lower bound for $P(\tau > n)$, for $1 \leqslant m \leqslant 7$ we have to evaluate the multivariate normal probabilities

$$\eta_{i,j} = P\Big\{\bigcap_{t=i}^{j} (S_t \in I_t)\Big\} \quad \text{for } i = j-6, \ldots, j-1,$$

where $I_t = (a_t, b_t)$ (equations (2.1)–(2.3)). For $j \geqslant i+1$ we condition on $S_{t}^{*}$, where $t^*$ is the integer part of $(i+j-1)/2$, and integrate over its density. By utilizing the conditional independence of the events a simplified expression for $\eta_{i,j}$ that is suitable for numerical integration is obtained. For example, to evaluate $\gamma_{7,n}$ the most elaborate term in equation (2.2) is

$$\eta_{j-6,j} = P\{\bigcap_{k=0}^{6} (S_{j-k} \in I_{j-k})\}.$$

Using the approach described above we get:

$$(3.1) \quad \eta_{j-6,j} = \int_{I_{j-3}} f_{S_{j-3}}(s) \prod_{k=0}^{1} P(S_{j-6+2k} \in I_{j-6+2k} \mid S_{j-5} = t,\ S_{j-3} = s) f_{S_{j-5}|S_{j-3}=s}(t)\, dt$$
$$\times \int_{I_{j-1}} \prod_{k=0}^{1} P(S_{j-2+2k} \in I_{j-2+2k} \mid S_{j-1} = u,\ S_{j-3} = s) f_{S_{j-1}|S_{j-3}=s}(u)\, du ds.$$

To evaluate the multiple integrals in equation (3.1) we have selected the Gaussian quadrature method. This method performs as well as other numerical integration methods and has the advantage of being based on a fixed number of points [22].

For the problem at hand we employ the Gauss–Legendre quadrature ([6], pp. 887 and 916–919):

$$(3.2) \qquad \int_{-a}^{a} f(x)\, dx = a \int_{-1}^{1} f(as)\, ds \approx \sum_{i=1}^{N} B_i [f(as_i) + f(-as_i)],$$

where $a > 0, f(x)$ is an integrable function, $\{B_i\}_{i=1}^{N}$ is a sequence of weights for the Gauss–Legendre quadrature, and $\{s_i\}_{i=1}^{N}$ is a sequence of abscissas (zeros of the $N$-th degree Legendre polynomial) for Gauss–Legendre quadrature ([6], pp. 916–919). The approximation on the right-hand side of equation (3.2) is based on $2N$ points. We now present the general form of the approximation for $\eta_{j-6,j}$ given in equation (3.1) that is based on the Gauss–Legendre quadrature. In what follows we assume without loss of generality that $I_t = (-c_t, c_t)$, where $c_t > 0$ and $t = 1, \ldots, n$. It follows from (3.1) that

$$(3.3) \quad \eta_{j-6,j} \approx \prod_{t=1}^{3} c_{k-t} \prod_{i=1}^{N} B_i \{a_i(s_i) \sum_{j=1}^{M} B_j g_j^{(1)}(s_i, t_j, -t_j) \sum_{h=1}^{M} B_h g_h^{(2)}(s_i, u_h, -u_h)$$
$$+ a_i(-s_i) \sum_{j=1}^{M} B_j g_j^{(1)}(-s_i, t_j, -t_j) \sum_{h=1}^{M} B_h g_h^{(2)}(-s_i, u_h, -u_h)\},$$

where $a_i(s_i)$, $g_j^{(1)}(s_i, t_j, -t_j)$, $g_j^{(1)}(-s_i, t_j, -t_j)$, $g_h^{(2)}(s_i, u_h, -u_h)$ and $g_h^{(2)}(-s_i, u_h, -u_h)$ are algebraic expressions involving the cumulative distributions and the density functions as functions of $\pm s_i$, $\pm t_i$ and $\pm u_i$, the abscissas used in the Gauss–Legendre quadrature. Explicit formulae for $\eta_{j-k,j}$, $k = 1, \ldots, 6$, are given in [10].

To evaluate the lower bound for $\mathrm{E}(\tau)$ and the approximation for $\mathrm{Var}(\tau)$ we substitute $\gamma_{7,n}$ for $P(\tau > n)$ in (1.6) and (1.7), respectively. Since the expressions for $\mathrm{E}(\tau)$ and $\mathrm{Var}(\tau)$ contain infinite series of tail probabilities, we truncate these series at the point where their values do not change by more than $10^{-6}$.

To approximate the power function, the $P$-values, the overall significance level, the median unbiased estimate of $\theta$, and the confidence interval for $\theta$ we effectively approximate the terms $\beta_n(\theta, x)$ given in (2.5) by $\gamma^*_{7,n}$ defined in (2.6). For $1 \leqslant m \leqslant 7$ the evaluation of $\gamma^*_{m,n}$ involves the terms $\gamma_{m,n-1}$ and $\gamma_{n-j,n-1}$ that were discussed above and the terms

$$\eta^*_{n-j,n} = P\{[\bigcap_{j=n-j}^{n-1} (S_j \in I_j)] \cap (S_n > b_n)\}, \qquad 1 \leqslant j \leqslant 6.$$

Since the evaluation of $\eta^*_{n-j,n}$ differs from $\eta_{n-j,n}$ only by the form of the last event, a similar numerical algorithm using the Gauss–Legendre quadrature is used. Explicit formulae for $\eta^*_{n-j,n}$, $1 \leqslant j \leqslant 6$, are given in [10].

The power function $\beta(\theta)$ and the overall significance level $\alpha$, given in (2.4) and (2.11), respectively, are represented as infinite series of $\beta_n(\theta, b_n)$. We truncate these series at the point where their values do not change by more than $10^{-6}$.

## 4. EXAMPLES

In this section we apply the approximations derived in Sections 2 and 3 to two well-established sequential tests.

EXAMPLE 1 (*sequential probability ratio test*). The continuation region for this test is given by the intervals

$$I_n = (-a/\theta_1 + n\theta_1/2,\ a/\theta_1 + n\theta_1/2),$$

where $a$ and $\theta_1$ are the design parameters. For more details on the sequential probability ratio test see [29], Chapter 3. To apply the approximations discussed in Sections 2 and 3 it is convenient to transform the continuation region so that it will be symmetric about the $n$ axis in the $(n, S_n)$-plane. This is accomplished by the transformation $Y_i = X_i - \theta/2$ and defining

$$\tau = \inf\{n \geqslant 1;\ S^*_n \notin I^*_n\},$$

where $S^*_n = \sum_{i=1}^{n} Y_i$ and $I^*_n = (-a/\theta_1, a/\theta_1)$. The null hypothesis is translated to $H_0$: $\theta \leqslant -\theta^*$ and the error rates requirements in this case are $\beta(-\theta^*) = 1 - \beta(\theta^*) = \alpha$, where $\theta^* = \theta_1/2$. To satisfy the above error requirements one uses the value $a = \ln(\gamma/\alpha) - 0.583$, where $\gamma$ is the Laplace transform of the asymptotic distribution of the excess of the random walk $\theta_1 S^*_n$ over the boundary evaluated at the value one (see [29], Section 3.1, and [21], Chapter X).

We now give the numerical results for this example. In Table I we present the approximations for the tail probabilities of the stopping time, in Table II the expected stopping time and the standard deviation of the stopping time, in Table III the power function of the test, in Table IV the $P$-values, and in Table V the approximate confidence intervals and the median unbiased estimators of $\theta$.

TABLE I. Lower bounds and simulated values of $P(\tau > n)$ for $\alpha = \beta = 0.05$ and $\theta^* = 0.50$

| $n$ | $\gamma_3$ | $\gamma_7$ | Simulation | $n$ | $\gamma_3$ | $\gamma_7$ | Simulation |
|---|---|---|---|---|---|---|---|
| 1 | 0.9706* | 0.9706* | 0.9667 | 11 | 0.0767 | 0.0826 | 0.0856 |
| 2 | 0.8273* | 0.8273* | 0.8276 | 12 | 0.0581 | 0.0636 | 0.0649 |
| 3 | 0.6592* | 0.6592* | 0.6633 | 13 | 0.0440 | 0.0489 | 0.0505 |
| 4 | 0.5130 | 0.5133* | 0.5112 | 14 | 0.0332 | 0.0376 | 0.0390 |
| 5 | 0.3951 | 0.3967* | 0.4000 | 15 | 0.0251 | 0.0290 | 0.0305 |
| 6 | 0.3026 | 0.3057* | 0.3153 | 16 | 0.0190 | 0.0223 | 0.0239 |
| 7 | 0.2309 | 0.2354* | 0.2386 | 17 | 0.0143 | 0.0172 | 0.0185 |
| 8 | 0.1757 | 0.1812 | 0.1865 | 18 | 0.0108 | 0.0132 | 0.0145 |
| 9 | 0.1335 | 0.1395 | 0.1405 | 19 | 0.0082 | 0.0102 | 0.0083 |
| 10 | 0.1013 | 0.1073 | 0.1098 | 20 | 0.0062 | 0.0078 | 0.0064 |

The starred values are exact.

TABLE II

| $\alpha\backslash\theta$ | $\theta^* = 0.25$ | | | $\theta^* = 0.50$ | | |
|---|---|---|---|---|---|---|
| | −0.25 | −0.125 | 0 | −0.50 | −0.25 | 0 |
| | Lower bounds and simulated values of $E(\tau)$ | | | | | |
| 0.010 | 31.39 | 46.22 | 59.46 | 9.05 | 14.13 | 19.04 |
| | 34.55 | 53.87 | 72.04 | 9.28 | 15.21 | 21.16 |
| | 36.16 | 61.31 | 84.90 | 9.23 | 15.25 | 21.54 |
| 0.050 | 19.51 | 25.77 | 29.53 | 5.62 | 7.70 | 8.99 |
| | 21.38 | 29.34 | 34.25 | 5.69 | 7.87 | 9.24 |
| | 22.32 | 30.58 | 36.20 | 5.72 | 7.99 | 9.25 |
| | Approximated and simulated values of standard deviation of $\tau$ | | | | | |
| 0.010 | 16.13 | 26.93 | 37.31 | 5.31 | 9.47 | 13.86 |
| | 19.82 | 35.51 | 51.17 | 5.68 | 11.10 | 16.78 |
| | 23.02 | 46.09 | 68.66 | 5.70 | 11.35 | 17.69 |
| 0.050 | 11.95 | 17.13 | 20.35 | 3.77 | 5.70 | 6.94 |
| | 14.53 | 21.77 | 26.35 | 3.92 | 6.02 | 7.37 |
| | 15.86 | 24.08 | 29.61 | 3.97 | 6.20 | 7.43 |

Upper and middle values are the lower bounds based on $\gamma_3$ and $\gamma_7$, respectively. Lower value is simulated.

TABLE III. Approximated and simulated values of $\beta(\theta)$

| $\alpha \backslash \theta$ | $\theta^* = 0.25$ −0.25 | −0.125 | 0 | $\theta^* = 0.50$ −0.50 | −0.25 | 0 |
|---|---|---|---|---|---|---|
| 0.010 | 0.0099 | 0.0906 | 0.5000 | 0.0099 | 0.0905 | 0.5000 |
| | 0.0099 | 0.0908 | 0.5000 | 0.0099 | 0.0908 | 0.5000 |
| | 0.0097 | 0.0886 | 0.5000* | 0.0103 | 0.0891 | 0.5000* |
| 0.050 | 0.0474 | 0.1823 | 0.5000 | 0.0475 | 0.1823 | 0.5000 |
| | 0.0476 | 0.1827 | 0.5000 | 0.0476 | 0.1826 | 0.5000 |
| | 0.0445 | 0.1798 | 0.5000* | 0.0488 | 0.1842 | 0.5000* |

Upper and middle values are the lower bounds based on $\gamma_3$ and $\gamma_7$, respectively. Lower value is simulated; starred values are exact.

TABLE IV. *P*-values for $\alpha = \beta = 0.05$ and $\theta^* = 0.25$

| $N$ | Upper boundary 0.1 | 0.5 | Lower boundary 0.1 | 0.5 |
|---|---|---|---|---|
| | Number of standard deviations of $S\tau$ over the boundary | | | |
| 1 | 0.0000 | 0.0000 | 1.0000 | 1.0000 |
| 2 | 0.0000 | 0.0000 | 0.9998 | 0.9999 |
| 3 | 0.0002 | 0.0001 | 0.9969 | 0.9984 |
| 4 | 0.0007 | 0.0004 | 0.9869 | 0.9914 |
| 5 | 0.0016 | 0.0012 | 0.9844 | 0.9849 |
| 6 | 0.0030 | 0.0024 | 0.9402 | 0.9512 |
| 7 | 0.0047 | 0.0039 | 0.9065 | 0.9196 |
| 8 | 0.0066 | 0.0057 | 0.8687 | 0.8831 |
| 9 | 0.0086 | 0.0077 | 0.8286 | 0.8438 |
| 10 | 0.0106 | 0.0097 | 0.7876 | 0.8030 |
| 11 | 0.0127 | 0.0118 | 0.7468 | 0.7621 |
| 12 | 0.0147 | 0.0138 | 0.7067 | 0.7216 |
| 13 | 0.0166 | 0.0158 | 0.6677 | 0.6822 |
| 14 | 0.0185 | 0.0177 | 0.6302 | 0.6441 |
| 15 | 0.0203 | 0.0195 | 0.5943 | 0.6076 |
| 16 | 0.0220 | 0.0213 | 0.5600 | 0.5727 |
| 17 | 0.0236 | 0.0229 | 0.5275 | 0.5395 |
| 18 | 0.0252 | 0.0245 | 0.4967 | 0.5081 |
| 19 | 0.0266 | 0.0260 | 0.4677 | 0.4784 |
| 20 | 0.0280 | 0.0274 | 0.4402 | 0.4503 |
| 21 | 0.0293 | 0.0287 | 0.4144 | 0.4239 |
| 22 | 0.0305 | 0.0300 | 0.3901 | 0.3991 |
| 23 | 0.0316 | 0.0311 | 0.3674 | 0.3758 |
| 24 | 0.0327 | 0.0322 | 0.3460 | 0.3539 |
| 25 | 0.0337 | 0.0333 | 0.3259 | 0.3333 |
| 26 | 0.0346 | 0.0342 | 0.3072 | 0.3141 |
| 27 | 0.0355 | 0.0351 | 0.2896 | 0.2961 |
| 28 | 0.0363 | 0.0360 | 0.2732 | 0.2792 |
| 29 | 0.0371 | 0.0368 | 0.2578 | 0.2634 |
| 30 | 0.0378 | 0.0375 | 0.2434 | 0.2487 |

TABLE V. 95% confidence intervals (CI) for $\theta$; $\alpha = \beta = 0.05$ and $\theta^* = 0.25$

| $N$ | Number of standard deviations $S\tau$ is over the upper boundary | | | | | |
|---|---|---|---|---|---|---|
| | 0.1 | | | 0.5 | | |
| | Lower CI | Median | Upper CI | Lower CI | Median | Upper CI |
| 1 | 3.5498 | 5.5097 | 7.4697 | 3.9498 | 5.9097 | 7.8697 |
| 2 | 1.3689 | 2.7549 | 4.1408 | 1.5689 | 2.9548 | 4.3407 |
| 3 | 0.7011 | 1.8342 | 2.9665 | 0.8287 | 1.9644 | 3.0977 |
| 4 | 0.5848 | 1.4689 | 1.7669 | 0.5848 | 1.4689 | 1.7669 |
| 5 | 0.2701 | 1.2562 | 1.6811 | 0.3574 | 1.3169 | 1.6990 |
| 6 | 0.1129 | 1.0271 | 1.6272 | 0.1619 | 1.0893 | 1.6373 |
| 7 | 0.0989 | 1.0078 | 1.6144 | 0.0989 | 1.0078 | 1.6144 |
| 8 | −0.0394 | 0.7218 | 1.5784 | 0.0106 | 0.8063 | 1.5860 |
| 9 | −0.0824 | 0.6321 | 1.5747 | −0.0227 | 0.7411 | 1.5833 |
| 10 | −0.1155 | 0.5624 | 1.5735 | −0.0504 | 0.6892 | 1.5825 |
| 11 | −0.1416 | 0.5064 | 1.5732 | −0.0733 | 0.6468 | 1.5822 |
| 12 | −0.1625 | 0.4604 | 1.5731 | −0.0925 | 0.6113 | 1.5822 |
| 13 | −0.1796 | 0.4218 | 1.5730 | −0.1086 | 0.5812 | 1.5822 |
| 14 | −0.1936 | 0.3889 | 1.5730 | −0.1224 | 0.5553 | 1.5821 |
| 15 | −0.2053 | 0.3607 | 1.5730 | −0.1341 | 0.5328 | 1.5821 |

EXAMPLE 2 (*asymptotically optimal Bayes sequential test*). The continuation region for this test is bounded and symmetric:

$$I_n = (-c_n + n\theta^*,\ c_n - n\theta^*),$$

where $c_n = (2an)^{1/2}$, $a > 0$, and $\theta^* > 0$. Moreover, $P(\tau \leqslant M) = 1$, where $M = 2a/\theta^{*2}$ if this quantity is an integer or the integer part of it plus one,

TABLE VI. Lower bounds and simulated values of $P(\tau > n)$ for $\alpha = \beta = 0.05$ and $\theta^* = 0.50$

| $n$ | $\gamma_3$ | $\gamma_7$ | Simulation | $n$ | $\gamma_3$ | $\gamma_7$ | Simulation |
|---|---|---|---|---|---|---|---|
| 1 | 0.9148* | 0.9148* | 0.9184 | 11 | 0.0978 | 0.1074 | 0.1067 |
| 2 | 0.8034* | 0.8034* | 0.8065 | 12 | 0.0709 | 0.0792 | 0.0771 |
| 3 | 0.6875* | 0.6875* | 0.6921 | 13 | 0.0503 | 0.0572 | 0.0551 |
| 4 | 0.5750 | 0.5765* | 0.5778 | 14 | 0.0349 | 0.0402 | 0.0377 |
| 5 | 0.4709 | 0.4749* | 0.4738 | 15 | 0.0235 | 0.0275 | 0.0266 |
| 6 | 0.3703 | 0.3850* | 0.3871 | 16 | 0.0153 | 0.0181 | 0.0183 |
| 7 | 0.2985 | 0.3075* | 0.3108 | 17 | 0.0095 | 0.0113 | 0.0109 |
| 8 | 0.2316 | 0.2419 | 0.2440 | 18 | 0.0055 | 0.0066 | 0.0069 |
| 9 | 0.1767 | 0.1875 | 0.1884 | 19 | 0.0029 | 0.0035 | 0.0034 |
| 10 | 0.1326 | 0.1431 | 0.1439 | 20 | 0.0013 | 0.0016 | 0.0014 |

The starred values are exact.

otherwise. The constants $a$ and $\theta^*$ are the design parameters for testing

$$H_0: \theta \leqslant -\theta^* \text{ vs. } H_a: \theta > -\theta^*,$$

and are selected in such a way that the error rates requirements in this case are $\beta(-\theta^*) = 1 - \beta(\theta^*) = \alpha$. A thorough discussion about this test can be found in [2], [19], and [28]. Numerical results for this example are presented in Tables VI–X.

TABLE VII

| $\alpha\backslash\theta$ | $\theta^* = 0.25$ | | | $\theta^* = 0.50$ | | |
|---|---|---|---|---|---|---|
| | $-0.25$ | $-0.125$ | 0 | $-0.50$ | $-0.25$ | 0 |
| | Lower bounds and simulated values of $E(\tau)$ | | | | | |
| 0.010 | 33.86 | 51.48 | 65.65 | 9.61 | 14.48 | 17.95 |
| | 37.75 | 58.29 | 73.77 | 9.93 | 15.11 | 18.66 |
| | 40.02 | 62.54 | 79.90 | 9.89 | 15.10 | 18.78 |
| 0.050 | 29.13 | 33.91 | 5.98 | 8.02 | 9.10 | |
| | 23.57 | 32.87 | 38.17 | 6.08 | 8.16 | 9.26 |
| | 24.30 | 34.62 | 40.36 | 6.09 | 8.12 | 9.33 |
| | Approximated and simulated values of standard deviation of $\tau$ | | | | | |
| 0.010 | 19.02 | 28.31 | 33.62 | 5.40 | 7.73 | 8.63 |
| | 22.09 | 32.45 | 37.08 | 5.74 | 8.13 | 8.91 |
| | 24.62 | 35.68 | 39.32 | 5.74 | 8.15 | 9.00 |
| 0.050 | 14.39 | 19.53 | 22.03 | 3.87 | 4.95 | 5.37 |
| | 16.66 | 22.21 | 24.58 | 3.98 | 5.07 | 5.47 |
| | 17.56 | 23.63 | 25.79 | 3.96 | 5.01 | 5.50 |

Upper and middle values are the lower bounds based on $\gamma_3$ and $\gamma_7$, respectively. Lower value is simulated.

TABLE VIII. Approximated and simulated values of $\beta(\theta)$

| $\alpha\backslash\theta$ | $\theta^* = 0.25$ | | | $\theta^* = 0.50$ | | |
|---|---|---|---|---|---|---|
| | $-0.25$ | $-0.125$ | 0 | $-0.50$ | $-0.25$ | 0 |
| 0.010 | 0.0083 | 0.0791 | 0.5000 | 0.0086 | 0.0954 | 0.5000 |
| | 0.0086 | 0.0871 | 0.5000 | 0.0090 | 0.1003 | 0.5000 |
| | 0.0086 | 0.0908 | 0.5000* | 0.0079 | 0.1031 | 0.5000* |
| 0.050 | 0.0413 | 0.1645 | 0.5000 | 0.0432 | 0.1805 | 0.5000 |
| | 0.0423 | 0.1706 | 0.5000 | 0.0439 | 0.1823 | 0.5000 |
| | 0.0471 | 0.1760 | 0.5000* | 0.0445 | 0.1822 | 0.5000* |

Upper and middle values are the lower bounds based on $\gamma_3$ and $\gamma_7$, respectively. Lower value is simulated; starred values are exact.

TABLE IX. $P$-values for $\alpha = \beta = 0.05$ and $\theta^* = 0.25$

| $N$ | Upper boundary | | Lower boundary | |
|---|---|---|---|---|
| | Number of standard deviations of $S\tau$ over the boundary | | | |
| | 0.1 | 0.5 | 0.1 | 0.5 |
| 1 | 0.0031 | 0.0009 | 0.9873 | 0.9958 |
| 2 | 0.0068 | 0.0051 | 0.9670 | 0.9764 |
| 3 | 0.0095 | 0.0083 | 0.9451 | 0.9547 |
| 4 | 0.0117 | 0.0107 | 0.9223 | 0.9321 |
| 5 | 0.0136 | 0.0127 | 0.9173 | 0.9182 |
| 6 | 0.0152 | 0.0145 | 0.8748 | 0.8847 |
| 7 | 0.0166 | 0.0160 | 0.8504 | 0.8603 |
| 8 | 0.0179 | 0.0173 | 0.8256 | 0.8356 |
| 9 | 0.0190 | 0.0185 | 0.8005 | 0.8105 |
| 10 | 0.0201 | 0.0196 | 0.7752 | 0.7852 |
| 11 | 0.0211 | 0.0206 | 0.7497 | 0.7597 |
| 12 | 0.0220 | 0.0216 | 0.7242 | 0.7342 |
| 13 | 0.0228 | 0.0224 | 0.6987 | 0.7086 |
| 14 | 0.0236 | 0.0233 | 0.6733 | 0.6831 |
| 15 | 0.0244 | 0.0241 | 0.6481 | 0.6578 |
| 16 | 0.0251 | 0.0248 | 0.6231 | 0.6327 |
| 17 | 0.0258 | 0.0255 | 0.5985 | 0.6079 |
| 18 | 0.0264 | 0.0262 | 0.5742 | 0.5834 |
| 19 | 0.0271 | 0.0268 | 0.5503 | 0.5594 |
| 20 | 0.0276 | 0.0274 | 0.2570 | 0.5358 |
| 21 | 0.0282 | 0.0280 | 0.5041 | 0.5127 |
| 22 | 0.0288 | 0.0285 | 0.4819 | 0.4902 |
| 23 | 0.0293 | 0.0290 | 0.4602 | 0.4683 |
| 24 | 0.0298 | 0.0296 | 0.4391 | 0.4470 |
| 25 | 0.0302 | 0.0300 | 0.4187 | 0.4263 |
| 26 | 0.0307 | 0.0305 | 0.3989 | 0.4063 |
| 27 | 0.0312 | 0.0310 | 0.3799 | 0.3869 |
| 28 | 0.0316 | 0.0314 | 0.3615 | 0.3683 |
| 29 | 0.0320 | 0.0318 | 0.3437 | 0.3503 |
| 30 | 0.0324 | 0.0322 | 0.3267 | 0.3330 |

From the numerical results for the two sequential tests given in Tables I–III and Tables VI–VIII, respectively, it is evident that the new approximations derived in this paper improve significantly on the approximations given in [9]. These improved approximations lead to accurate $P$-values and confidence intervals for the mean presented in Tables IV–V and IX–X. Note that the confidence intervals and the median unbiased estimators for the mean presented in Tables V and X were given only

TABLE X. 95% confidence intervals (CI) for $\theta$; $\alpha = \beta = 0.05$ and $\theta^* = 0.25$

| $N$ | Number of standard deviations $S_T$ is over the upper boundary | | | | | |
|---|---|---|---|---|---|---|
| | 0.1 | | | 0.5 | | |
| | Lower CI | Median | Upper CI | Lower CI | Median | Upper CI |
| 1 | 0.5261 | 2.4861 | 4.4461 | 0.9261 | 2.8861 | 4.8461 |
| 2 | 0.1557 | 1.6138 | 3.0209 | 0.2699 | 1.7803 | 3.2025 |
| 3 | 0.0137 | 1.2363 | 2.3956 | 0.0657 | 1.3305 | 2.5056 |
| 4 | −0.2670 | 0.6501 | 0.9591 | −0.2670 | 0.6501 | 0.9591 |
| 5 | −0.2793 | 0.5975 | 0.9019 | −0.2724 | 0.6187 | 0.9224 |
| 6 | −0.2911 | 0.5463 | 0.8475 | −0.2859 | 0.5644 | 0.8650 |
| 7 | −0.3032 | 0.5260 | 0.8207 | −0.3032 | 0.5260 | 0.8207 |
| 8 | −0.3129 | 0.4660 | 0.7551 | −0.3075 | 0.4907 | 0.7802 |
| 9 | −0.3145 | 0.4496 | 0.7390 | −0.3082 | 0.4812 | 0.7713 |
| 10 | −0.3152 | 0.4373 | 0.7288 | −0.3085 | 0.4742 | 0.7663 |
| 11 | −0.3157 | 0.4273 | 0.7229 | −0.3087 | 0.4685 | 0.7641 |
| 12 | −0.3159 | 0.4189 | 0.7200 | −0.3089 | 0.4638 | 0.7636 |
| 13 | −0.3161 | 0.4114 | 0.7193 | −0.3089 | 0.4598 | 0.7644 |
| 14 | −0.3163 | 0.4046 | 0.7202 | −0.3090 | 0.4562 | 0.7660 |
| 15 | −0.3164 | 0.3982 | 0.7223 | −0.3091 | 0.4530 | 0.7682 |

in the case when the process crosses the upper boundary. In the case when the lower boundary is crossed, for the examples presented in this paper, one multiples by the minus sign the values given in Tables V and X.

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